CMM3501
Advanced Mathematical Methods

Laplace Transforms III

Outline

❖ Finding the **matrix exponential** using **Laplace transforms**

❖ The **error function** (and related matters)

❖ Applications of **Laplace transforms** to the solution of **PDEs**

❖ Method of weighting functions **(for non-zero ICs)**

Matrix exponential (strategy)

$$e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$$

STEP 1: Given the matrix A, construct the new matrix $(sI - A)$.

STEP 2: The elements of the new matrix constructed are linear functions in s.
Find the inverse of this matrix by using standard matrix operations (see the background material posted on Brightspace). The elements of the inverse matrix will be fractions that depend on s.
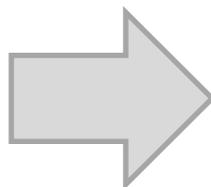
STEP 3: For each entry in the **inverse matrix** constructed at 'STEP 2', you will need to find the corresponding **inverse Laplace transform** (partial fractions and the simplified table will do the job in most cases). The inversion will produce functions of t.

STEP 4: Once you complete 'STEP 3', place all the results back into a matrix; that matrix will correspond to the matrix exponential on the left-hand side in the above formula.

$$e^{tA} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}$$

$$\mathbf{A} = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \qquad [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}$$

distinct eigenvalues 1 and 2

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s-5}{s^2 - 3s + 2} & \dfrac{6}{s^2 - 3s + 2} \\ \dfrac{-2}{s^2 - 3s + 2} & \dfrac{s+2}{s^2 - 3s + 2} \end{bmatrix}$$
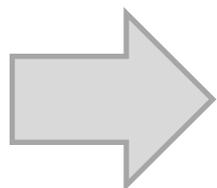
$$s^2 - 3s + 2 = (s - 2)(s - 1)$$

Matrix exponential (example 1 – cont'd)

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}$$

$$\frac{s-5}{s^2 - 3s + 2} = \frac{4}{s-1} - \frac{3}{s-2} \qquad\qquad \frac{6}{s^2 - 3s + 2} = \frac{6}{s-2} - \frac{6}{s-1} \qquad \text{ETC}$$
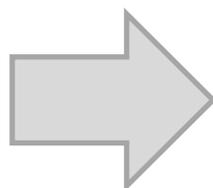
$$e^{t\mathbf{A}} = \begin{bmatrix} 4e^t - 3e^{2t} & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t} & -3e^t + 4e^{2t} \end{bmatrix}$$

(final answer)

Matrix exponential (example 2)

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}$$

$$\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}$$

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+3 & 4 \\ -2 & s-1 \end{bmatrix}$$
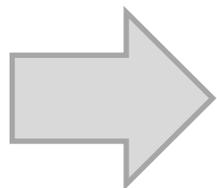
eigenvalues $-1 \pm 2i$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s-1}{s^2+2s+5} & \dfrac{-4}{s^2+2s+5} \\ \dfrac{2}{s^2+2s+5} & \dfrac{s+3}{s^2+2s+5} \end{bmatrix}$$

irreducible quadratic

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}$$

$$\frac{s-1}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$e^{t\mathbf{A}} = \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix}$$

(final answer)

ASIDE: the error function

$$\text{erf } s = \frac{2}{\sqrt{\pi}} \int_0^s \exp(-u^2)\,du$$

$$\text{erfc}(x) = 1 - \text{erf}(x),$$ **complementary error function**

These functions crop up in many places in applied mathematics.
For all practical purposes they should be regarded as being known.
They are available in most maths computer packages (e.g., MATLAB, MAPLE,etc).

Partial Differential Equations (PDEs)

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- Similar to ODEs, but the unknown function(s) depend on two or more independent variables.
- The ordinary differentiation is replaced by **partial differentiation** (hence the name).

$$H(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

most general form of
a **second-order PDE**
for a function of 2 independent variables

where H is an arbitrary function of its arguments, and for conciseness the suffix notation $u_x = \partial u/\partial x, u_y = \partial u/\partial y, u_{xx} = \partial^2 u/\partial x^2, u_{yx} = \partial^2 u/\partial x \partial y$, and $u_{yy} = \partial^2 u/\partial y^2$ has been used.

$$u_{xx} + u_{yy} = 0$$

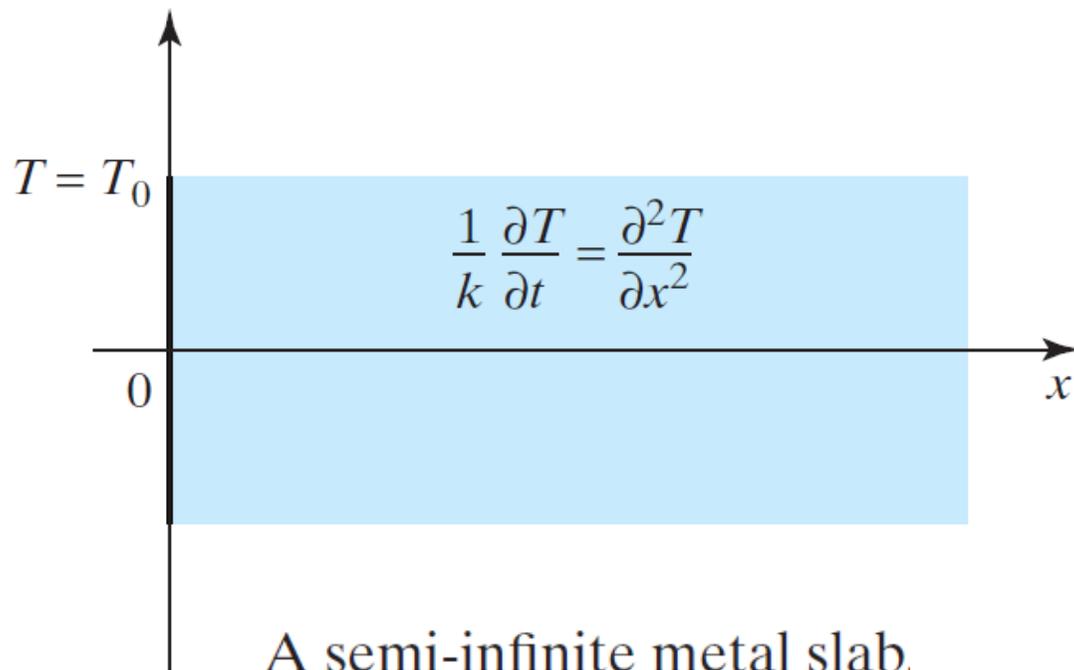$$w_{tt} - c^2 w_{xx} = 0.$$

$$u = u(x, y)$$

$$w = w(x, t)$$

(Laplace's equation)

(the wave equation)

$$\frac{1}{\kappa} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$
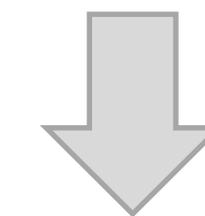
one-dimensional **heat equation**

$T = T(x, t)$ temperature in a heat-conducting solid at position x and time t

$$T(x, 0) = 0, \qquad x > 0$$
$$T(0, t) = T_0, \qquad t > 0$$

$$T(x, t) < \infty \qquad \text{for } x > 0, t > 0$$

$T = T_0$

$$\frac{1}{k} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

0

x

A semi-infinite metal slab.

$$T(x, t) = ?$$

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

OVERVIEW (solution strategy):

a) Take the Laplace transform w.r.t. t.
    That eliminates the partial derivative w.r.t. t
b) You are left with a 2nd order ODE in x
c) Use the given conditions (e.g., temperature must remain finite as $x \to \infty$) to simplify
    the general solution of the aforementioned 2nd order ODE.
d) Use the inverse Laplace transform to return back to the (x, t) domain
e) As a rule, the inversion in the previous step will invariably require a more complete
    inversion table than our simplified one. Such tables are available on Brightspace

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$
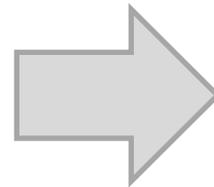
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$$\overline{T}(x,\,s) = {}_t\mathcal{L}\{T(x,\,t)\}$$

Laplace transform
w.r.t. t of the unknown
function

$${}_t\mathcal{L}\{\partial T(x,t)/\partial t\} = s\overline{T}(x,s) - T(x,0)$$

$${}_t\mathcal{L}\{\partial^2 T(x,t)/\partial x^2\} = \frac{\partial^2 \overline{T}(x,s)}{\partial x^2}$$

$$s\overline{T}(x,s) = \kappa\frac{d^2\overline{T}(x,s)}{dx^2}$$

$$\Longrightarrow \qquad \overline{T}'' - \frac{s}{\kappa}\overline{T} = 0$$

$$\overline{T}(x,s) = A\exp\left[\sqrt{\frac{s}{\kappa}}x\right] + B\exp\left[-\sqrt{\frac{s}{\kappa}}x\right]$$

$$\overline{T}(x, s) = B \exp\left[-\sqrt{\frac{s}{\kappa}}x\right]$$

$$T(0, t) = T_0 \qquad \Longrightarrow \qquad {}_t\mathcal{L}\{T(0, t)\} = T_0/s$$

$$\overline{T}(0, s)$$

$$\overline{T}(x, s) = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s}{\kappa}}x\right]$$

How do we Laplace invert this s-function??

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You can take the inverse from the more complete **table posted on Brightspace**.

$$\overline{T}(x, s) = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s}{\kappa}}x\right]$$

14	$\dfrac{a}{2\sqrt{\pi}}t^{-3/2}\,e^{-a^2/(4t)}$	$(a > 0)$	$e^{-a\sqrt{s}}$
15	$\mathrm{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$(a > 0)$	$\dfrac{1}{s}e^{-a\sqrt{s}}$
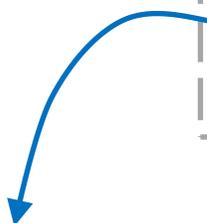
$$T(x, t) = T_0\,\mathrm{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\}, \qquad \text{for } x > 0, \ t > 0.$$

$$a = \frac{x}{\sqrt{k}}$$

$$\overline{T}(x, s) = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s}{\kappa}} x\right]$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$$

$$\frac{F(s)}{s}$$

14 $\quad \dfrac{a}{2\sqrt{\pi}} t^{-3/2} e^{-a^2/(4t)} \quad (a > 0)$

$e^{-a\sqrt{s}}$

$a = \dfrac{x}{\sqrt{k}}$

$$\overline{T}(x, s) = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s}{\kappa}} x\right]$$

The transform of an integral   Let $f(t)$ be a piecewise continuous function such that $|f(t)| \leq Me^{kt}$ for $k > 0$ and all $t \geq 0$. Then, if $\mathcal{L}\{f(t)\} = F(s)$,

(pp.406-407)

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \quad \text{for} \quad s > k,$$

$$\mathcal{L}^{-1}\{F(s)/s\} = \int_0^t f(\tau)d\tau.$$

$$\overline{T}(x,s) = \frac{T_0}{s} \exp\left[-\sqrt{\frac{s}{\kappa}}x\right]$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$$

$$\underbrace{\qquad\qquad\qquad}$$

$$\frac{F(s)}{s}$$

$$14 \quad \frac{a}{2\sqrt{\pi}}t^{-3/2}e^{-a^2/(4t)} \quad (a > 0) \qquad e^{-a\sqrt{s}} \qquad a = \frac{x}{\sqrt{k}}$$

$$T(x,t) = T_0 \int_0^t \frac{x t'^{-3/2}}{2\sqrt{k\pi}} \exp\left(-\frac{x^2}{4kt'}\right) dt' \longrightarrow T(x,t) = \frac{-2T_0}{\sqrt{\pi}} \int_\infty^{x/2\sqrt{kt}} \exp(-u^2) du$$

$$\frac{x}{2\sqrt{kt'}} = u \longrightarrow t'^{-3/2}dt' = -\frac{4\sqrt{k}}{x} du$$

$$= \frac{2T_0}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^\infty \exp(-u^2) du$$

$$T(x,t) = T_0 \int_0^t \frac{x t'^{-3/2}}{2\sqrt{k\pi}} \exp\left(-\frac{x^2}{4kt'}\right) dt' \longrightarrow \quad T(x,t) = -\frac{2T_0}{\sqrt{\pi}} \int_\infty^{x/2\sqrt{kt}} \exp(-u^2) du$$

$$= \frac{2T_0}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^\infty \exp(-u^2) du$$

(Gaussian integral)

$$\int_0^\infty \exp(-u^2)\, du = \frac{\sqrt{\pi}}{2} \quad \longrightarrow \quad \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-u^2)\, du = 1$$

$$\int_A^\infty (\dots) = \int_0^\infty (\dots) - \int_0^A (\dots) \quad \longrightarrow \quad T(x,t) = T_o\left(1 - \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{kt}} \exp(-u^2) du\right)$$

$$\frac{x}{2\sqrt{kt'}} = u \quad \longrightarrow \quad t'^{-3/2} dt' = -\frac{4\sqrt{k}}{x}\, du$$

$$T_0\left(1 - \mathrm{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right) = T_0\, \mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

Laplace transforms & PDEs

The **Laplace transforms** of $u(x, t)$ and its partial derivatives:

$$_t\mathcal{L}\{u(x, t)\} = U(x, s) = \int_0^\infty e^{-st} u(x, t) dt$$

$$_t\mathcal{L}\left\{\frac{\partial u(x, t)}{\partial t}\right\} = sU(x, s) - u(x, 0)$$

$$_x\mathcal{L}\left\{\frac{\partial u(x, t)}{\partial x}\right\} = sU(s, t) - u(0, t)$$

$$_t\mathcal{L}\left\{\frac{\partial^2 u(x, t)}{\partial t^2}\right\} = s^2 U(x, s) - su(x, 0) - u_t(x, 0)$$

$$_x\mathcal{L}\left\{\frac{\partial^2 u(x, t)}{\partial x^2}\right\} = s^2 U(s, t) - su(0, t) - u_x(0, t)$$

$$_t\mathcal{L}\left\{\frac{\partial^n u(x, t)}{\partial x^n}\right\} = \frac{d^n U(x, s)}{dx^n}, \quad n = 1, 2, \ldots.$$

Weighting function (Part 2)

We are going to look at the extension of the method discussed previously; this extension involves the case when the initial conditions are NOT zero.
This extension is briefly mentioned in **Q.25** on page 435 (in the chapter posted on Brightspace).
What you find below is all the details you need to apply the weighting function method in general.

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

$$y(0) = y_0, \ y^{(1)}(0) = y_1, \ y^{(2)}(0) = y_2, \dots, \ y^{(n-1)}(0) = y_{n-1}$$

By taking the Laplace transform of the equation and using the initial conditions:

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s) + H(s) = F(s)$$

$$P(s)$$

$$P(s)Y(s) + H(s) = F(s)$$

Matrix exponential

The weighting function:

$$a_n \frac{d^n w}{dt^n} + a_{n-1} \frac{d^{n-1} w}{dt^{n-1}} + \cdots + a_1 \frac{dw}{dt} + a_0 w = \delta(t)$$

$$w(0) = 0, \ w^{(1)}(0) = 0, \ w^{(2)}(0) = 0, \ldots, \ w^{(n-1)}(0) = 0$$

$$P(s)W(s) = 1 \ \Rightarrow \ W(s) = \frac{1}{P(s)} \qquad\qquad w(t) = \mathcal{L}^{-1}\{W(s)\}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$\downarrow$$

$$\delta(t), \delta'(t), etc$$

$$P(s)Y(s) + H(s) = F(s)$$

$$Y(s) = W(s)F(s) - W(s)H(s)$$

$$y(t) = \int_0^t w(\tau)f(t-\tau)\,d\tau \ - \ \int_0^t w(\tau)h(t-\tau)\,d\tau$$

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$$\delta(t) = \mathcal{L}^{-1}\{1\}$$

$$b\delta(t) = \mathcal{L}^{-1}\{b\}$$

$$\delta'(t) = \mathcal{L}^{-1}\{s\}$$

$$a\delta'(t) = \mathcal{L}^{-1}\{as\}$$

$$\mathcal{L}^{-1}\{As + B\} = A\delta'(t) + B\delta(t)$$

OBS.

In a 2nd order ODE, $H(s)$ will be at most a linear polynomial.
You'll need to use the above formula to get $h(t)$.

Example:

Solve the initial value problem

$$\begin{cases} y'' + 4y' - 5y = 8e^{3t} \\ y(0) = 5, \quad y'(0) = 0 \end{cases}$$

by using the weighting function method.

STEP 1: Find the weighting function:

$$w'' + 4w' - 5w = \delta(t) \qquad w(0) = w'(0) = 0$$

If $W(s) := \mathcal{L}[w(t)] \implies W(s) = \dfrac{1}{s^2 + 4s - 5} = \dfrac{1}{(s+5)(s-1)}$

$$w(t) = \mathcal{L}^{-1}\left[\frac{1}{6}\left(\frac{1}{s-1} - \frac{1}{s+5} \right) \right] = \frac{1}{6}\left(e^{t} - e^{-5t} \right)$$

So the weighting function is $\quad w(t) = \dfrac{1}{6}\left(e^{t} - e^{-5t} \right)$

STEP 2 : Identify $H(s)$ and then find $h(t) = \mathcal{L}^{-1}\{H(s)\}$

$$\mathcal{L}[y''] = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}[y'] = sY(s) - y(0)$$

$$Y(s) = \mathcal{L}[y(t)]$$

$$y(0) = 5 \qquad y'(0) = 0$$

Hence $H(s) = - sy(0) - 4y(0) = - 5s - 20$

$$h(t) = \mathcal{L}^{-1}[H(s)] = - 5 \mathcal{L}^{-1}[s] - 20 \mathcal{L}[1]$$
$$= - 5\delta'(t) - 20\delta(t)$$

$$h(t) = - 5\delta'(t) - 20\delta(t)$$

STEP 3 : We are now ready to apply the formula :

$$y(t) = \int_0^t w(\tau) f(t-\tau) d\tau - \int_0^t w(\tau) h(t-\tau) d\tau$$

$$= \underbrace{\int_0^t \frac{1}{6} \left(e^{\tau} - e^{-5\tau} \right) 8 e^{3(t-\tau)} d\tau}_{J_1} - \underbrace{\int_0^t \frac{1}{6} \left(e^{\tau} - e^{-5\tau} \right) \left[-5\delta'(t-\tau) - 20\delta(t-\tau) \right] d\tau}_{J_2}$$

$$J_1 = \frac{4}{3} \int_0^t \left(e^{\tau} \cdot e^{3t-3\tau} - e^{-5\tau} \cdot e^{3t-3\tau} \right) d\tau$$

$$= \frac{4}{3} \int_0^t \left(e^{3t-2\tau} - e^{3t-8\tau} \right) d\tau = \frac{4}{3} e^{3t} \int_0^t \left(e^{-2\tau} - e^{-8\tau} \right) d\tau$$

$$= \frac{4}{3} e^{3t} \left[\left(-\frac{1}{2} e^{-2\tau} \right) \Big|_0^t + \left(\frac{1}{8} e^{-8\tau} \right) \Big|_0^t \right]$$

$$= \frac{4 e^{3t}}{3} \left[-\frac{1}{2} e^{-2t} + \frac{1}{2} + \frac{1}{8} e^{-8t} - \frac{1}{8} \right]$$

$$\boxed{J_1 = -\frac{2e^t}{3} + \frac{1}{2} e^{3t} + \frac{1}{6} e^{-5t}}$$

$$\boxed{\delta'(-x) = -\delta'(x)} \qquad \boxed{\int f(\tau)\,\delta'(\tau-t)\,d\tau = -f'(t)}$$

discussed in the previous session

$$J_2 = \frac{5}{6}\int_0^t (e^\tau - e^{-5\tau})\,\delta'(\tau-t)\,d\tau \;-\; \frac{10}{3}\int_0^t (e^\tau - e^{-5\tau})\,\delta(\tau-t)\,d\tau$$

$$= \frac{5}{6}\left(-(e^t + 5e^{-5t})\right) - \frac{10}{3}\left(e^t - e^{-5t}\right)$$

$$J_1 - J_2 = -\frac{2}{3}e^t + \frac{1}{2}e^{3t} + \frac{1}{6}e^{-5t} + \frac{5}{6}e^t + \frac{25}{6}e^{-5t} + \frac{10}{3}e^t - \frac{10}{3}e^{-5t}$$

$$= \left(-\frac{2}{3} + \frac{5}{6} + \frac{10}{3}\right)e^t + \frac{1}{2}e^{3t} + \left(\frac{1}{6} + \frac{25}{6} - \frac{10}{3}\right)e^{-5t}$$

$$= \frac{21}{6}e^t + \frac{1}{2}e^{3t} + \frac{6}{6}e^{-5t}$$

$$\boxed{y(t) = \frac{7}{2}e^t + e^{-5t} + \frac{1}{2}e^{3t}} \quad \leftarrow \text{ required solution}$$